ON A METHOD OF SOLUTION OF INTEGRAL EQUATIONS WITH LARGE VALUES OF THE PARAMETER

(OB ODNOM SPOSOBE RESHENIIA INTEGRAL'NYKH URAVNENII PRI BOL'SHOM ZNACHENII PARAMETRA)

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In engineering problems if frequently becomes necessary to solve the nonhomogeneous integral equation

$$y = \lambda K y + f \tag{1}$$

when the absolute value of the parameter λ is considerably larger than the absolute value of the first characteristic value. We will assume that the function f can be expanded as a series in terms of the characteristic functions of the homogeneous equation:

$$f = \sum_{n=1}^{\infty} c_n y_n \qquad \left(c_n = (f,\ y_n) = \int_a^b f y_n h dx\right) \tag{2}$$

The characteristic functions y_n are orthogonalized and normalized with a weight function $h(x)$ over the interval $(a,\ b)$:

$$(y_n,\ y_m) = \int_a^b y_n y_m h dx = \begin{cases} 0 & (n \neq m) \\ 1 & (n = m) \end{cases}$$

In order to obtain the solution of equation (1) in the form of the infinite series

$$y = \sum_{n=1}^{\infty} \frac{c_n y_n}{1 - \lambda/\lambda_n} \tag{3}$$

it is necessary to determine a large number of characteristic functions y_n and the corresponding characteristic constants λ_n, which may involve considerable difficulties.

It is for this reason that the methods of successive approximations have found wide application.

The method of simple iteration for the equation (1) will converge only when $|\lambda| < |\lambda_1|$. For large values of λ it becomes necessary to use more complicated iteration processes [1], which, however, cease to be effective if $|\lambda| \gg |\lambda_1|$, (in practical cases $|\lambda| > 10\,|\lambda_1|$).

Below is given a method of solution of equation (1) which makes it possible to apply a converging method of simple iteration after a certain finite number of characteristic functions have been found.*

For the purpose of explaining the principle of the method, let us first suppose that the parameter λ is less than the second characteristic value (i.e. $|\lambda_1| < |\lambda| < |\lambda_2|$). In place of equation (1) we consider the transformed equation [1]

$$y = \lambda K_2 y + f \tag{4}$$

where

$$K_2 y = Ky - y_1 \int_a^b K y y_1 h\,dx = Ky - y_1 (Ky,\ y_1)$$

We will solve this equation by the method of simple iteration according to the scheme

$$y_{(i)} = \lambda K_2 y_{(i-1)} + f \qquad (i = 1, 2, \ldots) \tag{5}$$

Let us suppose that the initial $y_{(0)}$ can be represented in the form

$$y_{(0)} = \sum_{n=1}^{\infty} c_n y_n \tag{6}$$

Substituting this expression in equation (5), we obtain

$$y_{(1)} = c_1 y_1 + \sum_{n=2}^{\infty} c_n y_n + \sum_{n=2}^{\infty} \frac{\lambda}{\lambda_n} c_n y_n$$

and

$$y_{(2)} = c_1 y_1 + \sum_{n=2}^{\infty} c_n \left(1 + \frac{\lambda}{\lambda_n}\right) y_n + \sum_{n=2}^{\infty} \frac{\lambda^2}{\lambda_n^2} c_n y_n$$

$$y_{(3)} = c_1 y_1 + \sum_{n=2}^{\infty} c_n \left(1 + \frac{\lambda}{\lambda_n} + \frac{\lambda^2}{\lambda_n^2}\right) y_n + \sum_{n=2}^{\infty} \frac{\lambda^3}{\lambda_n^3} c_n y_n, \ldots$$

* Independently of the author, A.F. Gurov, in his dissertation defended in 1957, used a method similar in principle to the one given here, on a problem on the vibration of a shaft.

As i increases, the approximations converge to the function

$$y^* = c_1 y_1 + \sum_{n=2}^{\infty} \frac{c_n y_n}{1 - \lambda/\lambda_n} \tag{7}$$

which differs from the exact solution only by the first term.

The exact solution of equation (1) can now be obtained in the form

$$y = y^* + \frac{c_1 y_1}{\lambda_1/\lambda - 1} \tag{8}$$

Taking into account equation (3) we obtain

$$y = y^* + \frac{y_1}{\lambda_1/\lambda - 1} \int_a^b f y_1 h dx \tag{9}$$

When λ lies between the characteristic values λ_{j-1} and λ_j (i.e. when $|\lambda_{j-1}| < |\lambda| < |\lambda_j|$), we solve the equation

$$y = \lambda K_j y + f \qquad \left(K_j y = Ky - \sum_{n=1}^{j-1} y_n (Ky,\ y_n)\right) \tag{10}$$

by the method of simple iteration.

If y^* is a solution of equation (10), the solution of the original integral equation takes the form

$$y = y^* + \sum_{n=1}^{j-1} \frac{y_n}{\lambda_n/\lambda - 1} \int_a^b f y_n h dx \tag{11}$$

In practice, solving the problem should begin with the successive determination of characteristic functions and characteristic values, until one finds a λ_j such that $|\lambda_j| > |\lambda|$. After this equation (10) is solved by the usual method of iteration.

The method presented is easily extended to matrix integral equations [1]

$$[y] = \lambda [K^{(\nu)}] [y] + [f]$$

for which the condition of orthogonality and normalization takes the form

$$([y_n],\ [y_m])^{(\nu)} = \int_a^b \sum_{r=0}^{\nu} y_n^{(r)} y_m^{(r)} h_r dx = \begin{cases} 0 & (n \neq m) \\ 1 & (n = m) \end{cases}$$

where $y_n^{(r)}$ and $y_m^{(r)}$ are r-th-order derivatives of the characteristic functions y_n and y_m respectively.

BIBLIOGRAPHY

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